Asymmetric dynamics and critical behavior in the Bak-Sneppen model

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Abstract

We investigate, using mean-field theory and simulation, the effect of asymmetry on the critical behavior and probability density of Bak-Sneppen models. Two kinds of anisotropy are investigated: (i) different numbers of sites to the left and right of the central (minimum) site are updated and (ii) sites to the left and right of the central site are renewed in different ways. Of particular interest is the crossover from symmetric to asymmetric scaling for weakly asymmetric dynamics, and the collapse of data with different numbers of updated sites but the same degree of asymmetry. All non-symmetric rules studied fall, independent of the degree of asymmetry, in the same universality class. Conversely, symmetric variants reproduce the exponents of the original model. Our results confirm the existence of two symmetry-based universality classes for extremal dynamics.

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I. INTRODUCTION

Nature exhibits scale invariance in a variety of settings. Such behavior is often associated with power law distributions of the phenomenon of interest, for example earthquake sizes [1,2], rain intensities and drought durations [3–5] and physiological and morphological quantities [6,7]. In recent decades, physicists have sought the physical origin of such laws [2,5,7]. A concept introduced to partially explain the ubiquity of scale-invariant phenomena in nature is so-called *self-organized criticality* or SOC [2,8].

The Bak-Sneppen (BS) model [8,9] was proposed to explain mass extinctions observed in the fossil record, and has attracted much attention as a prototype of SOC under extremal dynamics [10]. The model has been studied through various approaches, including simulation [11–14], theoretical analysis [15–17], probabilistic analysis (run time statistics) [18,19], renormalization group [20,21], field theory [22] and mean-field theory [9,10,23–27]; it was recently adapted to model experimental data on bacterial populations [28,29]. Applications of the model in evolution studies are reviewed in [30]. Asymmetric versions of the model were studied in Refs. [31] and [32]. In this paper, we study how varying the degree of symmetry affects the stationary distribution and the critical behavior, using mean-field theory and numerical simulation.

In the evolutionary interpretation of the BS model, each site i represents a biological species, and bears a real-valued variable x_i representing its "fitness". The larger x_i, the better adapted this species is to its environment and so the more likely it is to survive. At each time step, the site with the smallest x_i and its nearest neighbors are replaced with randomly chosen values. The replacement of x_i represents extinction of the less-fit species and the appearance of a new one, while the substitution of the neighboring variables with new random values may be interpreted as a sudden unpredictable change in fitness when an interdependent species goes extinct and a new species colonizes its niche. Selection, at each step, of the global minimum of the $\{x_i\}$ ("extremal dynamics") represents a highly nonlocal process, and would appear to require an external agent with complete information regarding the state of the system at each moment.

Due to the extremal dynamics, this system exhibits scale-invariance in the stationary state, in which several quantities display power-law behavior [8]. Simulations show that the stationary distribution of barriers follows a step function, being zero (in the infinite-size limit) for $x < x^* \simeq 0.66702(3)$ [12]. Relaxing the extremal condition leads to a smooth probability density and loss of scale invariance [10,26,27]. Datta et. al have also shown that the scaling behavior is sensitive to the number of sites updated at each step (i.e., updating only the minimum site, or the minimum and next-to-minimum as well) [33].

In this paper we investigate the effect of symmetry in the updating rule. Section II introduces the models, which are then analyzed using mean-field like approaches in Sec. III. In Sec. IV we present simulation results; our conclusions are summarized in Sec. V.

II. MODELS

The Bak-Sneppen model [8] is defined as follows. Consider a d-dimensional lattice with L^d sites and periodic boundaries. In the evolutionary interpretation of the model, each site i represents a species, and bears a real-valued variable $x_i(t)$ representing its "fitness" to survive, so that $x_i(t)$ may be termed a "barrier to extinction". The initial values of the barriers are independently drawn from a uniform distribution on the interval [0,1). At time 1, the site m bearing the minimum of all the numbers $\{x_i(0)\}$ is identified, and it, along with its $2d$ nearest neighbors, are given new random values, again drawn independently from the interval [0,1). (In the one dimensional case considered here this amounts to: $x_m(1) = \eta$, $x_{m+1}(1) = \eta'$, and $x_{m-1}(1) = \eta''$, where η, η', and η'' are independent and uniformly distributed on [0,1); for $|j-m| > 1$, $x_j(1) = x_j(0)$.) At step 2 this process is repeated, with m representing the site with the global minimum of the variables $\{x_i(1)\}$, and so on.

We now define several one-dimensional variants of the BS model, which differ from the original in the number and/or position of neighbors which are updated at each time step, or in the way that the barriers x_i evolve. In the 'generalized' or 'BSab' variant, we replace the site m bearing the minimum of the $\{x_i\}$ plus a neighbors on the left side and b neighbors on the right with independent random numbers. If $a = b = 1$ we recover the original model (BS11); if $a = 0$ and $b = 1$ we have the anisotropic BS model (BS01) studied in [31,32]; if $a \neq b$ we obtain modified BS models with asymmetric dynamics. These BSab variants of the Bak-Sneppen model were also studied in [31]. In the second variant, the site M bearing the *maximum* of the $\{x_i\}$ and its two nearest neighbors are updated according to the rules: $x_M(t+1) = \eta$, $x_{M'}(t+1) = \eta'$ and $x_{M''}(t+1) = [x_{M''}(t)]^2$. Here $M' = M + \sigma$ and $M'' = M - \sigma$, where σ is $+1$ with probability p and -1 otherwise. We shall refer to this variant as the peripheral square model with variable anisotropy. (For $p = 1/2$ this is the peripheral square model studied in [27].)

The motivation for studying these variants is threefold. First it is of interest to examine the effect of (i) the symmetry of the dynamics and (ii) replacement of barriers with a deterministic function $f(x)$ instead of random numbers, on the critical behavior of the model. Secondly, we study corrections to the power-laws, such as finite-size effects, and find that these corrections are large for small asymmetries in agreement with [31]. Finally, since the precise form of the dynamics in a specific setting (e.g., evolution) is generally unknown, and probably is quite different from that of the original model, it is of interest to test the robustness of the results reported for the original model. It is even possible that some of the variants considered approximate a given evolutionary process more closely than the original. In particular, if certain pairs of species (i and j, say) stand in a predator-prey relationship, one would not expect the extinction of i to have the same effect on j as the extinction of j has on i; in this situation an asymmetric interaction appears more reasonable.

III. MEAN-FIELD THEORY

We develop a mean-field theory for the BSab variants, along the lines of Refs. [10,26]. To begin, we introduce a flipping rate of $\Gamma e^{-\beta x_i}$ at site i, where Γ^{-1} is a characteristic time, irrelevant to stationary properties, and which we set equal to one ($\Gamma = 1$). This regularized system is the 'finite-temperature' model [10,26,27], in which all sites have a nonzero probability of being updated at any time, in contrast to the extremal dynamics of the original model. The extremal condition is recovered in the zero-temperature limit, $\beta \to \infty$; a regularized flipping rate facilitates analysis of the model.

The probability density $p(x,t)$ satisfies:

$$\frac{dp(x,t)}{dt} = -e^{-\beta x}p(x,t) - \sum_{j=1}^{a}\int_0^1 e^{-\beta y}p_j(x,y,t)dy - \sum_{j=1}^{b}\int_0^1 e^{-\beta y}p_j(x,y,t)dy + n\int_0^1 e^{-\beta y}p(y,t)dy, \tag{1}$$

where $n = a + b + 1$ is the number of sites that are updated at each step, $p_j(x,y,t)$ is the joint density for sites 0 and j and $p(y,t)$ is the one-site marginal density. (We assume translation and reflection invariance, which is expected to hold at any time, if the initial distribution possesses these properties.) Invoking the mean-field factorization $p_j(x,y,t) = p(x,t)p(y,t)$, we find:

$$\frac{dp(x,t)}{dt} = -p(x,t)[e^{-\beta x} + (n-1)I(\beta)] + nI(\beta) \quad , \tag{2}$$

where

$$I(\beta) \equiv \int_0^1 e^{-\beta y}p(y,t)dy \tag{3}$$

represents the overall flipping rate. In the stationary state we have

$$p_{st}(x) = \frac{nI}{(n-1)I + e^{-\beta x}} \quad . \tag{4}$$

Multiplying by $e^{-\beta x}$ and integrating over the range of x, we find $I(\beta) = (e^{(n-1)\beta/n} - 1)/[(n-1)e^{\beta}(1 - e^{-\beta/n})]$ and thus

$$p_{st}(x) = \frac{n}{(n-1)}\frac{1 - e^{-(n-1)\beta/n}}{1 - e^{-(n-1)\beta/n} + e^{-\beta x}(e^{\beta/n} - 1)} \quad . \tag{5}$$

In the limit $\beta \to \infty$ this solution becomes a step function:

$$p_{st}(x) = \frac{n}{(n-1)}\Theta(x - 1/n)\Theta(1 - x) \quad . \tag{6}$$

Thus, the mean-field approach predicts a step-function singularity for the probability density with the critical barrier at $x^* = 1/n$. This result is independent

of the symmetry of the dynamics, and we conclude that the mean-field approximation at the level of the one-site marginal density is insensitive to differences in symmetry.

Define the anisotropy coefficient

$$k_a = \frac{|a-b|}{n}. \tag{7}$$

(Note that $k_a = 0$ for symmetric dynamics.) The mean-field threshold $x^* = 1/n$ can be written as

$$x^* = \frac{1}{(2a+1)}(1-k_a) \quad \text{for } b > a. \tag{8}$$

For fixed a, mean-field theory predicts that the threshold x^* varies linearly with the anisotropy coefficient k_a.

IV. SIMULATIONS

A. Threshold values

We study the generalized model for various values of a and b, corresponding to k_a in the range $0-0.857$. We estimate the probability density $p(x)$ on the basis of a histogram of barrier frequencies, dividing [0,1] into 100 subintervals. After the system (a ring of $N = 2000$ sites) relaxes to the stationary state[1], histograms are accumulated at intervals of N time steps until a total of 10^8 steps are performed.

The threshold x^*_N for a finite system can be calculated as follows [34]. In the limit $N \to \infty$, the probability density $p(x)$ is a step function, with $p(x) = 0$ for $x < x^*$ and $p(x) = C$ for $x > x^*$, where C is a constant. Normalization implies that $C(1-x^*) = 1$. This suggests that $x^*_N = 1 - 1/C_N$ can be regarded as the threshold of a finite system. In ref. [34] we performed simulations for various systems sizes and found that $x^*_N - x^*_\infty = kN^{-1/\nu}$, with $x^*_\infty = 0.6672(2)$ and $\nu = 1.40(1)$ in the original model (or BS11) and $x^*_\infty = 0.7240(1)$ and $\nu = 1.58(1)$ in the anisotropic BS model (or BS01). Typically $x^*_N - x^*_\infty \approx 0.006$ for $N = 2000$ and, therefore, the threshold $x^*_{N=2000}$ is sufficiently accurate for our present purpose, which is to investigate the effect of anisotropy on the threshold values.

Simulation confirm that the stationary probability density is a step function, in agreement with MFT. Fig. 1 shows $p(x)$ for all variants having $n = 5$. The threshold values, listed in Table I, are however always larger than that predicted by MFT, $x^* = 1/n$. MFT is exact for the random-neighbor version,

[1]Typically, the system can be considered to be in the stationary state after $\approx 10^7$ time steps.

in which all sites are considered neighbors, and in which $(n-1)$ randomly selected sites are updated in addition to the site with the minimum x_i. There is better agreement between the MFT prediction for x^* and the simulation result for larger anisotropies (with n constant), and for larger n (with fixed k_a). MFT predicts that $x^* = x^*(n)$, whereas in fact, for fixed n, the threshold is *smaller*, the larger the anisotropy. This may be understood in general terms as follows. With n fixed, the larger $|a-b|$, the greater is the distance between updated sites and the minimum site. This makes the dynamics resemble the mean-field (random neighbor) case more closely, causing the threshold to tend toward the MFT value $1/n$. Increasing n (with k_a fixed) should have a similar effect.

Figure 2 shows that if we fix a and vary b, the threshold decreases linearly with k_a, as predicted by MFT, except for $a=0$. The actual threshold values are, however, quite different from MFT results. For instance for $a=1$, MFT predicts $x^* = (1/3)(1-k_a)$, while a fit to the simulation data yields $x^* = 0.664(3) - 0.668(7)k_a$.

B. Critical exponents

Several quantities display power-law behavior in the Bak-Sneppen model, and can be used to characterize the associated universality class [8,11,31]. In particular, we study the probability $P_J(r)$ that successive updated sites are separated by a distance r. In the original model, one finds [31]

$$P_J(r) \sim r^{-\pi}, \quad \text{with} \quad \pi = \pi_S = 3.23(2) \tag{9}$$

(figures in parentheses denote uncertainties). On the other hand, in the anisotropic BS model (BS01), $\pi = \pi_A = 2.401(2)$ [32].

Based on simulations of the generalized models using 7×10^8 time steps on lattices of 32000 sites we find that all variants with asymmetric dynamics belong to the universality class of the anisotropic BS model. Fig. 3 displays $P_J(r)$ for variants with different anisotropies. The corrections to the power laws are large for small asymmetries and therefore the asymptotic behavior is not observed in small lattices in these cases, because large values of r are not reached.

We find that $P_J(r)$ can be fitted with an expression representing a crossover between symmetric scaling at small r and asymmetric scaling at large r,

$$P_J(r) = A[r^{-\pi_A} + (N-r)^{-\pi_A}] + B[r^{-\pi_S} + (N-r)^{-\pi_S}] \quad , \tag{10}$$

where A and B are constants and π_A and π_S are, respectively, the exponents of the anisotropic and the original models quoted above. The 'mirror' structure of Eq. (10) arises due to the periodic boundary conditions, which imply that $P_J(r) = P_J(N-r)$. Eq. (10) provides a good fit to the simulation data for $r > 10^2$ for all generalized models. Since $\pi_S \approx \pi_A + 1$, the expression $P_J(r) = Ar^{-\pi_A}(1+B/r) + A(N-r)^{-\pi_A}(1+B/(N-r))$ also fits the data

well. Nevertheless, studies of weakly asymmetric models (e.g., BS(20)(21), for which $k_a = 0.024$) indicate that the slope of $P_J(r)$ (on log scales) approaches π_S, as opposed to $\pi_A + 1$, before the asymptotic behavior is attained.

The functions $P_J(r)$ for variants with the same anisotropy can be collapsed onto a master curve

$$P_{collapse}(k_a, r') = n \; P_J(r/n) \quad , \tag{11}$$

where $r' = r/n$. Figure 4 illustrates this collapse for five variants with $k_a = 1/6$. A simple argument provides an intuitive understanding of this scaling function. Suppose site m is the extremal site at time t. Since renewed sites receive independent random numbers, all sites in the set $\mathcal{N}(t) \equiv \{m(t) - a, m(t) - a + 1, ..., m(t) + b\}$ have the same probability of being the extremum at time $t + 1$, which implies (with $a \leq b$)

$$2P_J(0) = P_J(1) = P_J(2) = ... = P_J(a), \tag{12}$$

so that $P_J(r)$ exhibits a plateau for $r = 1, ..., a$.

The probability of event $m(t+1) \in \mathcal{N}(t)$, i.e., the extremal site belongs to the group of sites updated at the last time step, can be estimated as follows. We assume that, with a probability ≈ 1, all sites outside of $\mathcal{N}(t)$ have $x_i > x^*$. Then the probability that $m(t+1) \in \mathcal{N}(t)$ is $1 - (1 - x^*)^n$. In MFT we then have $P(1) = \frac{2}{n}[1 - (1 - x^*)^n] \propto \frac{1}{n}$, consistent with the prefactor n in Eq. (11). (Observe that for the original BS model, $1 - (1 - x^*)^n \simeq 0.96$.)

If $b > a$, then

$$P_J(a + 1) \simeq P_J(a + 2) \simeq \cdots \simeq P_J(b) \simeq \frac{1}{2} P_J(1) \tag{13}$$

where the symbol '$\simeq$' is used because there is a small probability of one of the distances $r = a+1, ..., b$ being the extremum distance even if $m(t+1) \notin \mathcal{N}(t)$. This explains the second plateau seen in Fig. 4. For fixed k_a, $a = [n(1 - k_a) - 1]/2$ and $b = [n(1 + k_a) - 1]/2$ are essentially proportional to n, so that the rescaling of the argument in Eq. (11) collapses the plateaus. For $r > b$ the probability P_J rapidly approaches a power law, which is also invariant under the rescaling of Eq. (11). Although the foregoing arguments are approximate, they provide an intuitive basis for the collapse seen in Fig. 4.

Another quantity that obeys a power law in the Bak-Sneppen model is $P_r(\tau)$, the probability that, in the stationary state, the global minimum site at time t was the extremal site most recently at time $t - \tau$. (τ is the 'return time.') We simulate the BSab variants on a lattice of 16000 sites, using 7×10^8 time steps. The results are quite similar to those for the probability $P_J(r)$. For weak anisotropies, the asymptotic behavior is observed only for large τ, as shown in Fig. 5. In the limit $\tau \to \infty$, $P_r(\tau) \propto \tau^{-b}$, where $b = 1.40(1)$ for all anisotropic variants and $b = 1.58(1)$ for all isotropic cases.

C. Peripheral square model with variable anisotropy

The peripheral square model with variable anisotropy, defined in Sec. II, is a generalization of the model studied in [27], where the anisotropy now depends on a parameter p. We simulated this model (with $N = 2000, 4000$ and 8000 sites) for p varying from the isotropic value ($p = 0.5$) to the maximally anisotropic one ($p = 1.0$). Our analysis shows that, in the limit $L \to \infty$, the stationary probability density consists of two regions (see Fig. 6): $p_{st}(x) = 0$ for $x > x^*$ and $p_{st}(x) \neq 0$ for $x < x^*$. (The rounding in the threshold region is simply a finite-size effect.) This behavior is in agreement with Ref. [27], where we found that renewing the barriers via $x' = x^\alpha$, with $\alpha = 1/2$ and 2, leads to a diversity of distributions with a discontinuity at the threshold. The divergence of p_{st} as $x \to 0$ in the present case can be understood on the basis of the mean-field theory developed in [27]. We conclude that the step-function distribution of the original BS model is not universal for self-organized criticality under extremal dynamics. In contrast, our results suggest that the step-singularity is universal.

Figure 6 suggests that, in the limit $x \to 0$, $p_{st}(x)$ does not depend on the parameter p. Over restricted intervals p_{st} appears to follow a power law. For example, for $x \in [10^{-6}, 10^{-4}]$, p_{st} is well described by a power law with an exponent that increases from 0.82 for $p = 0.5$ to 0.85 for $p = 1.0$. On the other hand, the mean-field solution, which appears to capture the qualitative behavior, does not follow a power law, diverging instead as $\sum_{n=0}^{\infty} 4^{-n} x^{2^{-n}-1}$ in the limit $x \to 0$. (We further note that although the MF solution accompanies the general trend of the data, it exhibits a series of step singularities in addition to the jump at x^*. Several step discontinuities are in fact observed in p_{st} in simulations of the random-neighbor version [27], although not in the nearest-neighbor version.)

The dependence of the threshold x^* on p is shown in Fig. 7. Note that augmenting the anisotropy, x^* increases almost linearly. Therefore, similarly to the BSab variant, the effect of increasing the anisotropy, while maintaining the number of updated sites constant, is to decrease the interval on which $p(x) = 0$.

Figure 8 shows the probability density $P_J(r)$ for various values of p. While the isotropic case ($p = 0.5$) preserves the exponent of the original BS model, all cases for $p \neq 0.5$ exhibit the exponent of the anisotropic BS model. In the BSab variant studied above, asymmetry was due to different numbers of sites being renewed in each side of the extremal site. In the present case we have a different kind of asymmetry, namely, sites are updated in different ways on each side of the extremal one. We nevertheless find the same critical exponents as for the anisotropic BS model. This strengthens the evidence for the existence of two symmetry-based universality classes for models under extremal dynamics.

V. CONCLUSIONS

We perform a detailed investigation of the effect of symmetry on the scaling behavior of the Bak-Sneppen model. All dynamics which preserve the reflection symmetry of the original BS model possess the same critical exponents as the original model, while asymmetric dynamics lead to the exponents of the anisotropic BS model. Therefore, our work reinforces the evidence for two symmetry-based universality classes [31].

In order to obtain these results, we define modified BS models, and study them via mean-field theory and simulation. In the generalized BS model (or BSab variant), the degree of asymmetry is quantified by the anisotropy coefficient k_a. Mean-field theory provides poor predictions for the threshold x^*, but correctly predicts (i) that the stationary probability density is a step function, and (ii) that x^* varies linearly with the number n of updated sites, as found in simulations on a one-dimensional lattice. The simulations also lead to several conclusions that go beyond MFT analysis: (i) the threshold value decreases as the degree of anisotropy is increased; (ii) for weak anisotropy we observe a crossover between symmetric and asymmetric scaling; (iii) for fixed anisotropy we find a scaling collapse of the probability $P_J(r)$ that successive minimum sites have a separation r.

In the peripheral square model with variable anisotropy, the degree of asymmetry is quantified by the parameter p that controls on which side of the extremal site the neighbor is updated differently. Although this model is partially deterministic and has a different kind of asymmetry, we again encounter two symmetry-based universality classes. Moreover, we find that increasing the asymmetry, the region where $p(x) = 0$ is reduced, as in the the BSab variant.

These results, together with evidence for finite-size scaling [34], and connections with directed percolation [22], indicate that scaling in the BS model partakes of many of the characteristics associated with critical phenomena, both in and out of equilibrium. The particular distinguishing features of the Bak-Sneppen model and its variants appear to be associated with extremal dynamics.

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TABLES

TABLE I. Threshold (x^*_N) for systems of $N = 2000$ sites for generalized BSab models with various anisotropy coefficients (k_a). Figures in parentheses denote uncertainties.

MODEL	$x^*_{N=2000}$	k_a
BS01	0.717(2)	1/2
BS02	0.531(2)	2/3
BS03	0.417(2)	3/4
BS04	0.342(2)	4/5
BS05	0.290(2)	5/6
BS06	0.251(2)	6/7
BS11	0.661(9)	0
BS12	0.501(3)	1/4
BS13	0.398(2)	2/5
BS14	0.329(3)	1/2
BS15	0.280(2)	4/7
BS22	0.466(9)	0
BS23	0.381(3)	1/6
BS24	0.317(3)	2/7
BS25	0.271(2)	3/8
BS26	0.237(2)	4/9
BS27	0.210(2)	1/2
BS33	0.36(1)	0

FIGURES

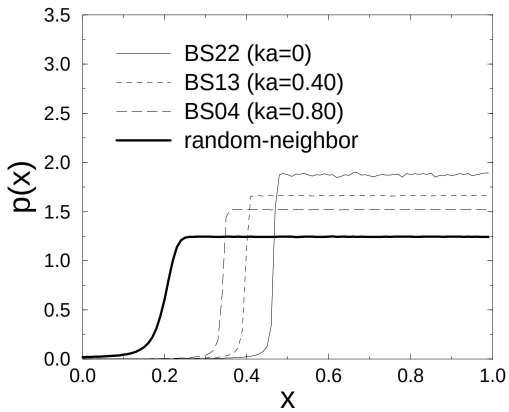

FIG. 1. Stationary probability densities for the BSab with $n = 5$.

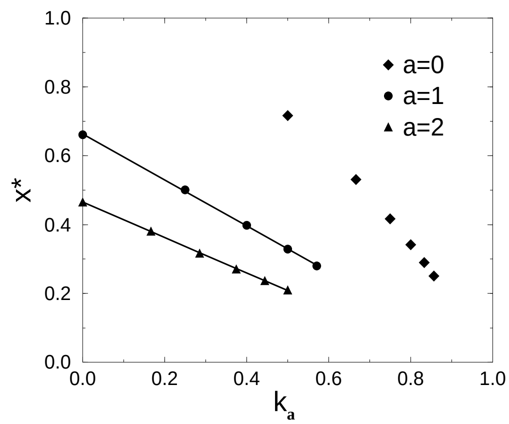

FIG. 2. Threshold values as a function of the anisotropy coefficient k_a. We fix a and change b to vary k_a. Further explanation in text.

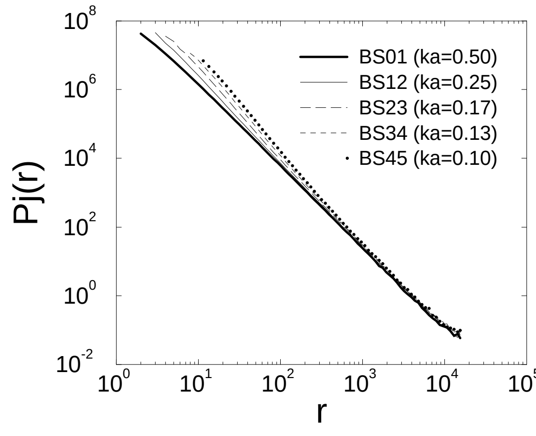

FIG. 3. Stationary probability $P_J(r)$ for various BSab variants.

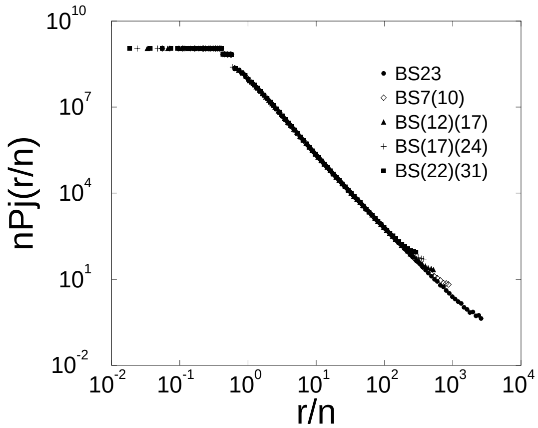

FIG. 4. Scaling collapse of $P_J(r)$ for variants with $k_a = 1/6$, see Eq. (11).

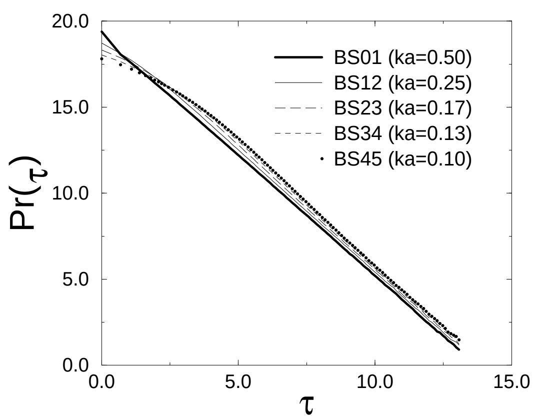

FIG. 5. Stationary probability $P_r(\tau)$ for BSab variants.

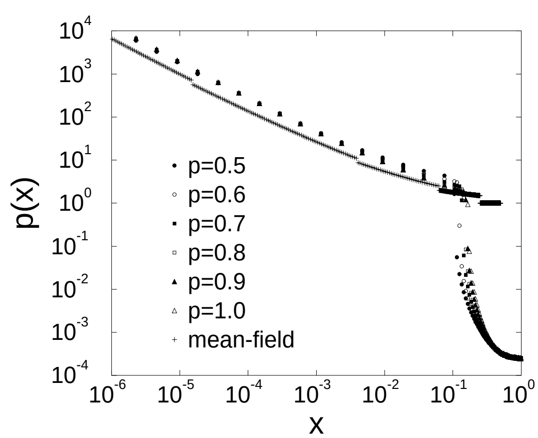

FIG. 6. Stationary probability density $p(x)$ given by mean-field theory and simulation ($N = 8000$ sites). The parameter p varies from the symmetric value (p=0.5) to the maximaly asymmetric one (p=1.0).

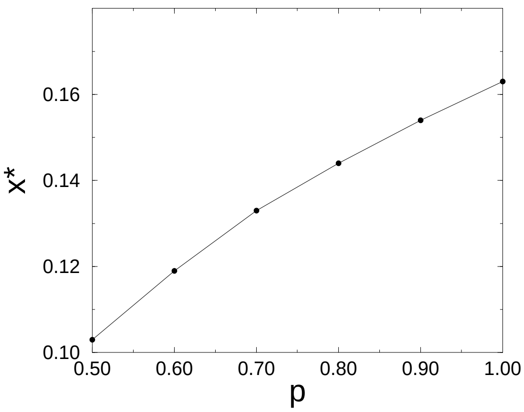

FIG. 7. Thresholds of the peripheral square models as a function of the parameter p.

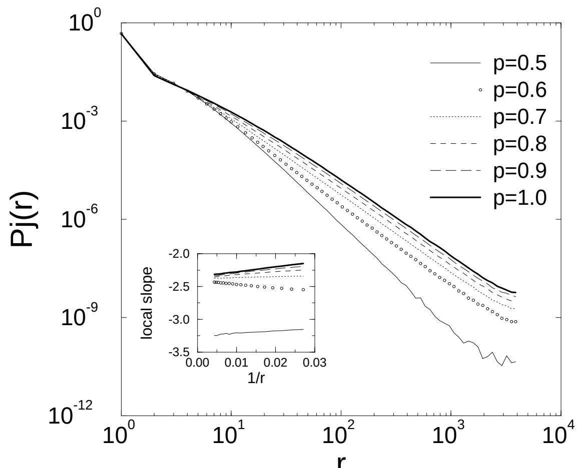

FIG. 8. Stationary probability $P_J(r)$ for the peripheral square model with different values of p. Inset: Local slope *vs.* $1/r$. (Note that the slope - exponent π - approaches $-\pi_A = -2.401$ in the limit $r \to \infty$ in all cases with $p \neq 0.5$.)